# ECON 402 Discussion: Week 2 (lecture)

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# Announcements

- Quiz 1 available on Canvas at 4pm today!

- Discussion (lecture): Fridays at 11am in AH G127 (recordings after).
- Discussion (problems): Fridays at 2pm in AH G127 (recordings after).
- Office hours: Thursdays after 3pm (email ahead), Fridays at 3pm standing, or by appointment anytime!

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- Topics today
  1. Introduction
  2. Consumer choice
  3. Uncertainty and risk

# Introduction

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- Today, we endogenize (bring into the model) the saving rate with a classic utility maximizing model of consumption and saving.
- Our goal is to provide microfoundations for aggregate consumption $C_t$ and investment $I_t$, and learn about their evolution.

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- Today, we endogenize (bring into the model) the saving rate with a classic utility maximizing model of consumption and saving.
- Our goal is to provide microfoundations for aggregate consumption $C_t$ and investment $I_t$, and learn about their evolution.

- Setting: A single consumer who lives for $T \geq 2$ periods, starts life with $A_1 \geq 0$ endowment, earns income stream $\{y_t\}_{t=1}^{T}$, and chooses consumption and saving $\{c_t, s_t\}_{t=1}^{T}$ to maximize lifetime utility $U$.

# Consumer Choice

- Example: If there are $T = 2$ periods, derive the lifetime budget constraint from the period budget constraints

$$
\begin{aligned}
t = 1 &\quad : \quad c_1 + s_1 = y_1 + A_1 \\
t = 2 &\quad : \quad c_2 = (1 + r)s_1 + y_2
\end{aligned}
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where $(c_t, s_t, y_t, A_t)$ denotes consumption, saving (or borrowing), income, and assets at time $t$.

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- Definition: With $T \geq 2$ periods, the present value lifetime budget constraint is

$$c_1 + \frac{1}{1+r}c_2 + \cdots + \left(\frac{1}{1+r}\right)^{T-1} = A_1 + y_1 + \frac{1}{1+r}y_2 + \cdots + \left(\frac{1}{1+r}\right)^{T-1} y_T$$

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- Definition: The "Euler equation" in any given period $t$ is a necessary condition for an agent to be optimally choosing their consumption $c_t$ (and hence saving $s_t$) levels.
  - Given our assumptions on utility, it is sufficient too!
  - To solve any dynamic consumer problem, need to find this object.
  - It's got a fancy name, but it's just a FOC...

# Consumer Choice

- Example: If a consumer has time-separable log utility with discount factor $\beta \in (0, 1)$ given by

$$\begin{aligned} U &:= \ln c_1 + \beta \ln c_2 + \cdots + \beta^{T-1} \ln c_T \\ &= \sum_{t=1}^{T} \beta^{t-1} \ln c_t \end{aligned}$$

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- Solution: We derive the FOC to the consumer problem

$$\begin{aligned} \max_{c_1,\ldots,c_T} \quad & \sum_{t=1}^{T} \beta^{t-1} \ln c_t \\ \text{s.t.} \quad & \sum_{t=1}^{T} \left(\frac{1}{1+r}\right)^{t-1} c_t = A_1 + \sum_{t=1}^{T} \left(\frac{1}{1+r}\right)^{t-1} y_t \end{aligned}$$

using the Lagrangian ("grown up") and MRS condition methods.

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$$\mathscr{L} = \sum_{t=1}^{T} \beta^{t-1} \ln c_t + \lambda \left[ A_1 + \sum_{t=1}^{T} \left( \frac{1}{1+r} \right)^{t-1} y_t - \sum_{t=1}^{T} \left( \frac{1}{1+r} \right)^{t-1} c_t \right]$$

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$$\mathscr{L}_{c_{t+1}} : \beta^{t} \frac{1}{c_{t+1}} + \lambda \left( - \left( \frac{1}{1+r} \right)^{t} \right) = 0$$

# Consumer Choice

1. Lagrange method: We can solve the two FOCs

$$\lambda = \frac{\beta^{t-1}\frac{1}{c_t}}{\left(\frac{1}{1+r}\right)^{t-1}} = \frac{\beta^{t}\frac{1}{c_{t+1}}}{\left(\frac{1}{1+r}\right)^{t}}$$

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$$\frac{c_{t+1}}{c_t} = \beta(1+r)$$

which gives the consumer's Euler equation!

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$$MRS_{t,t+1} \quad := \quad \frac{\frac{\partial}{\partial c_t} U}{\frac{\partial}{\partial c_{t+1}} U} = \frac{\beta^{t-1} \frac{1}{c_t}}{\beta^t \frac{1}{c_{t+1}}} = \frac{c_{t+1}}{\beta c_t}$$

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\Rightarrow \frac{c_{t+1}}{c_t} &= \beta(1 + r)
\end{aligned}
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which *again* gives the consumer's Euler equation!

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- Note: The Euler equation

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  - $\beta \uparrow$ (more patient) $\Rightarrow$ $G_c \uparrow$
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- Note: When combined with the lifetime budget constraint

$$\sum_{t=1}^{T} \left(\frac{1}{1+r}\right)^{t-1} c_t = A_1 + \sum_{t=1}^{T} \left(\frac{1}{1+r}\right)^{t-1} y_t$$

the Euler equation gives an exact solution for $\{c_t^*, s_t^*\}_{t=1}^{T}$.

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- Definition: Let $X$ be some random variable with probability mass function (discrete) or probability distribution function (continuous) given by $f_X(x)$. The expected value of $X$ is

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\begin{aligned}
E(X) &:= \sum_x x f_X(x) \qquad \text{if discrete} \\
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- Definition: The variance of $X$ is given by

$$Var(X) := E(X^2) - E(X)^2$$

and measures the dispersion of values of $X$ around its mean $E(X)$. The standard deviation is $sd(X) := \sqrt{Var(X)}$.

# Uncertainty and Risk

- Note: If $X_t$ is a random variable measured at time $t$, then its expected value today is simply $E_t(X_t) = X_t$ whereas its expectation tomorrow

$$E_t(X_{t+1})$$

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- Example: Let $\varepsilon_t \sim \text{UNIF}[-1, 1]$ be a random shock and suppose that income evolves (stochastically) according to a "random walk", so

$$Y_{t+1} := Y_t + \varepsilon_{t+1}$$

with $Y_0$ given. Compute $E_t(Y_{t+1})$ and $E_2(Y_{t+1})$.

# Uncertainty and Risk

- Part a: Since the random shock is uniform between $[-1, 1]$, we know that $E(\varepsilon_t) = 0$ in every time period (you will show this in homework!). We substitute the process for income to find

$$\begin{aligned} E_t(Y_{t+1}) &= E_t(Y_t + \varepsilon_{t+1}) \\ &= E_t(Y_t) + E_t(\varepsilon_{t+1}) \\ &= Y_t + 0 = Y_t \end{aligned}$$

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$$

- Part b: We similarly compute

$$
\begin{aligned}
E_2(Y_{t+1}) &= E_2(Y_t + \varepsilon_{t+1}) \\
&= E_2(Y_{t-1} + \varepsilon_t + \varepsilon_{t+1}) \\
&= ... \\
&= E_2(Y_1 + \varepsilon_2 + \cdots + \varepsilon_{t+1}) \\
&= Y_1 + \varepsilon_2 + 0 + \cdots + 0 \\
&= Y_1 + \varepsilon_2
\end{aligned}
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- Solution: Let $X$ be the income you get if you choose the gamble. Using the fact that the coin is fair, we compute

$$E(X) = \sum_{x} x f_X(x) = 8 \cdot 0.5 + 0 \cdot 0.5 = 4$$

and then find utility under each scenario

$$u(E(X)) = \sqrt{4} = 2$$

$$E(u(X)) = u(8) \cdot 0.5 + u(0) \cdot 0.5 = \sqrt{8} \cdot 0.5 + \sqrt{0} \cdot 0.5 = 1.4$$

# Uncertainty and Risk

- Theorem: The utility function $u(c)$ represents preferences which are risk-averse if and only if $u''(c) < 0$.
- Example: With $u(c) = \sqrt{c} = c^{\frac{1}{2}}$, we have

$$\begin{aligned} u'(c) &= \frac{1}{2}c^{-\frac{1}{2}} > 0 \\ u''(c) &= -\frac{1}{4}c^{-\frac{3}{2}} < 0 \end{aligned}$$

  which shows our gambler was risk-averse before.

- Note: We usually assume consumers are risk-averse. How does this all relate to the neoclassical assumptions we made about a firm's production function?