ECON 402 Discussion: Week 2 (problems)

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Announcements

- Quiz 1 available on Canvas at 4pm today!

- Discussion (lecture): Fridays at 11am in AH G127 (recordings after).
- Discussion (problems): Fridays at 2pm in AH G127 (recordings after).
- Office hours: Thursdays after 3pm (email ahead), Fridays at 3pm standing, or by appointment anytime!

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- Topics today
  1. Practice exam solutions
  2. Interest rates and welfare

Practice exam solutions

#1 (Consumption) Suppose an agent lives for $T = 2$ periods with income stream $(y_1, y_2) = (20, 10)$, discount factor $\beta = 1$, and lifetime utility function

$$u(c_1, c_2) = \ln(c_1) + \beta \ln(c_2)$$

where c_t denotes consumption in period t and $r \geq 0$ is the interest rate.

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The consumer's optimization problem is given by

$$\max_{c_1, c_2} [\ln(c_1) + 1 \cdot \ln(c_2)] \qquad s.t \qquad c_1 + \frac{1}{1+0} c_2 = 20 + \frac{1}{1+0} 10$$

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We can use the optimality condition to get the Euler equation

$$MRS = \frac{p_1}{p_2} \quad \Leftrightarrow \quad \frac{\frac{1}{c_1}}{\frac{1}{c_2}} = \frac{1}{1} \quad \Leftrightarrow \quad c_1 = c_2$$

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which we can combine with the lifetime budget constraint to get consumption

$$c_1 + c_2 = 30 \quad \Leftrightarrow \quad c_1 + c_1 = 30$$

which implies that $c_1^* = c_2^* = 15$, so there is perfect consumption smoothing.

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which implies that $c_1^* = c_2^* = 15$, so there is perfect consumption smoothing. This is enabled by saving $s_1^* = y_1 - c_1^* = 20 - 15 = 5$ in the first period. Lifetime utility in this case is $U_0^* = u(c_1^*, c_2^*) = \ln(15) + \ln(15) \approx 5.42$

Practice exam solutions

b) Find optimal consumption c_t^* and saving s_t^* $\forall t$ when the interest rate is $r = 1$. Compute lifetime utility $U_1^* := u(c_1^*, c_2^*)$ in this case and compare to U_0^*. Under which interest rate is the consumer better off; does this make sense?

b) Find optimal consumption c_t^* and saving s_t^* $\forall t$ when the interest rate is $r = 1$. Compute lifetime utility $U_1^* := u(c_1^*, c_2^*)$ in this case and compare to U_0^*. Under which interest rate is the consumer better off; does this make sense?
The consumer's optimization problem is given by

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We can use the optimality condition to get the Euler equation

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which we can combine with the lifetime budget constraint to get consumption

$$c_1 + \frac{1}{2} c_2 = 25 \quad \Leftrightarrow \quad c_1 + \frac{1}{2} 2c_1 = 25$$

which implies that $c_1^* = 12.5$ and $c_2^* = 25$, so there is no longer perfect consumption smoothing. This is enabled by saving $s_1^* = y_1 - c_1^* = 20 - 12.5 = 7.5$. Lifetime utility is $U_1^* = \ln(12.5) + \ln(25) \approx 5.74$. The consumer is better off under the higher interest rate because they are savers and can earn a greater return on their savings in this case.

Practice exam solutions

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When $r = 0$, we see that being rich earlier or later in life doesn't matter. The solution is identical to part a, with lifetime utility $\tilde{U}_0^* = U_0^* \approx 5.42$. However, when the interest rate is $r = 1$, we can see that the consumer will be a borrower in their attempts to smooth consumption over time. They should be worse off relative to part c; we find

$$\tilde{c}_1^* = 10 \qquad \tilde{c}_2^* = 20 \qquad \tilde{s}_1^* = 0 \qquad \tilde{U}_1^* \approx 5.30$$

so the consumer chooses not to borrow and instead consumes their endowment of income.

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d) Now assume that $r = 0$ and $(y_1, y_2) = (20, 10)$ again, but the consumer faces a borrowing constraint such that they cannot borrow more than 5 units of the consumption good. Solve for optimal consumption and saving in this case, and compute lifetime utility $\tilde{U}_0^*$. Compare this to U_0^*; does this make sense.

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We previously showed that when the interest rate is zero and the consumer is rich early in life, they will be a saver to perfectly smooth consumption. Thus, the borrowing constraint $s_1 \geq -5$ will not bind, since they will choose $s_1^* > 0$. Lifetime utility is identical to part a.

Practice exam solutions

#2 (Production Functions and Growth Accounting) Suppose an economy is characterized by the aggregate production function $Y_t = F(K_t, L_t)$ where Y_t is total output, K_t is the capital input, and L_t is the labor input, or number of workers.

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a) If the production function takes the constant elasticity of substitution (CES) form given by $F(K_t, L_t) = A(\alpha K_t^r + (1-\alpha)L_t^r)^{\frac{1}{r}}$ where $A > 0$, $\alpha \in (0,1)$, and $r < 1$, show that both marginal products are positive and decreasing.

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$$MP_L \quad = \quad \frac{\partial}{\partial L_t} F = \frac{A}{r}(\alpha K_t^r + (1-\alpha)L_t^r)^{\frac{1-r}{r}} \cdot (1-\alpha) r L_t^{r-1} > 0$$

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\frac{\partial}{\partial L_t} MP_L &= A(1-\alpha) \cdot [(\alpha K_t^r + (1-\alpha)L_t^r)^{\frac{1-r}{r}} (r-1) L_t^{r-2} + \dots \\
&\quad \dots L_t^{r-1} \frac{1-r}{r} (\alpha K_t^r + (1-\alpha)L_t^r)^{\frac{1-2r}{r}} (1-\alpha) r L_t^{r-1}] < 0
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b) For the CES production function above, show that there are constant returns to scale.

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b) For the CES production function above, show that there are constant returns to scale.

$$
F(\lambda K_t, \lambda L_t) = A(\alpha(\lambda K_t)^r + (1-\alpha)(\lambda L_t)^r)^{\frac{1}{r}} = A(\lambda^r)^{\frac{1}{r}} (\alpha K_t^r + (1-\alpha) L_t^r)^{\frac{1}{r}} = \lambda F(\lambda K_t, \lambda L_t)
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c) It can be shown that the CES production function approaches the Cobb-Douglas production function $F(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha}$ as $r \to 0$. Find the wage rate and the rental rate of capital under the assumption that inputs are chosen to maximize profits.

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$$w_t = MP_L = A(1-\alpha)K_t^{\alpha} L_t^{-\alpha}$$
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Practice exam solutions

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d) With the Cobb-Douglas production function above, write down the growth rate of total output Y_t as a function of the growth rates of capital g_K and labor n.

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e) Under the assumptions of the Solow growth model with Cobb-Douglas production, write down the growth rate of consumption per capita as a function of g_K and n.

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$$c_t = (1-s)y_t$$
$$g_c = (1-s)g_y = (1-s)\cdot\alpha(g_K - n)$$

Practice exam solutions

#3 (Solow Growth Model) Suppose that the economy is characterized by the standard assumptions of the Solow growth model, but that production is Cobb-Douglas in capital and *effective* labor. Specifically, total output is given by

$$Y_t = K_t^{\alpha} L_t^{1-\alpha} = K_t^{\alpha}(E_t N_t)^{1-\alpha}$$

where K_t is capital, L_t is effective labor, E_t is technology, or the productivity of a given worker, and N_t is the total number of workers. Assume these grow at rates g_K, g_E, and n.

a) Does this production function exhibit constant returns to scale (CRS) in capital and workers? Prove your answer using the definition of CRS.

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a) Does this production function exhibit constant returns to scale (CRS) in capital and workers? Prove your answer using the definition of CRS.
The production function is Cobb-Douglas and exhibits constant returns to scale in capital and labor. We can prove this by assuming $\lambda > 0$ is a constant and computing

$$\begin{aligned} F(\lambda K_t, \lambda N_t) &= (\lambda K_t)^{\alpha}(E_t \lambda N_t)^{1-\alpha} \\ &= \lambda^{\alpha} K_t^{\alpha} \lambda^{1-\alpha}(E_t N_t)^{1-\alpha} \\ &= \lambda K_t^{\alpha}(E_t N_t)^{1-\alpha} \\ &= \lambda F(K_t, N_t) \end{aligned}$$

Practice exam solutions

b) Write down output per effective worker $y_t := \frac{Y_t}{E_t N_t}$ as a function of capital per effective worker $k_t := \frac{K_t}{E_t N_t}$ in this economy.

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c) Let $s \in (0, 1)$ denote the exogenous saving rate and $\delta \in (0, 1)$ denote the depreciation rate of capital. Using the law of motion of the total capital stock K_t, derive the law of motion of capital per effective worker k_t.

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$$\begin{aligned} \frac{\partial}{\partial s}k_* &= \frac{\partial}{\partial s}\left(\frac{s}{\delta + g_E + n}\right)^{\frac{1}{1-\alpha}} \\ &= \left(\frac{1}{\delta + g_E + n}\right)^{\frac{1}{1-\alpha}} \frac{1}{1-\alpha} s^{\frac{1}{1-\alpha}-1} \\ &= \left(\frac{1}{\delta + g_E + n}\right)^{\frac{1}{1-\alpha}} \frac{1}{1-\alpha} s^{\frac{\alpha}{1-\alpha}} \\ &> 0 \end{aligned}$$

Practice exam solutions

e) In the steady state, find the growth rate of capital per effective worker k_* from above, capital per individual worker $\tilde{k}_*$ where $\tilde{k}_t := \frac{K_t}{N_t}$, and total output Y_t.

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In steady state, we know that $g_{k_*} = 0$ so that $g_{\tilde{k}_*} = g_E$. For output, $y_t = \frac{Y_t}{E_T N_t}$ implies that $Y_t = y_t \cdot E_t N_t$. Taking logs and differentiating with respect to time, we have

$$g_Y = g_y + g_E + n$$

Since $y_* = k_*^\alpha$ is constant in the steady-state, its growth rate is zero. Thus, the growth rate of total output is equal to the sum of technology growth and population growth

$$g_{Y_*} = g_E + n$$

Interest Rates and Welfare

- Example: Let $T = 2$, $\beta = 1$, and $r \geq 0$. Given utility and income

$$\begin{aligned} u(c_1, c_2) &= \ln c_1 + \ln c_2 \\ (y_1, y_2) &= (2, 4) \end{aligned}$$

answer the following questions.

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Interest Rates and Welfare

- Part b: Find optimal consumption and saving when $r = 1$.

  Using the same strategy as before, we find that $c_1^{**} = 2$, $c_2^{**} = 4$, and $s_1^{**} = y_1 - c_1^{**} = 2 - 2 = -1$. Since $s_1^{**} = 0$, our consumer neither borrows nor saves, they simply eat what they have!

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$$
\begin{aligned}
r = 0 \quad &: \quad U_0 = \ln c_1^* + \ln c_2^* \approx 2.19 \\
r = 1 \quad &: \quad U_1 = \ln c_1^{**} + \ln c_2^{**} \approx 2.07
\end{aligned}
$$

  It makes sense that the consumer is worse off under a higher interest rate, since they were a borrower before the change, and borrowing became more expensive!

- Part d: Assume $r = 0$ again, but the consumer faces a borrowing constraint $s_1 > -\bar{s} = -0.5$. Find utility if they consume optimally.

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  That $U_0^{\bar{s}} < U_0 \approx 2.19$ is unsurprising, since the consumer is constrained when they want to borrow more!