ECON 402 Discussion: Week 4 (lecture)

Elird Haxhiu

University of Michigan

haxhiu@umich.edu

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Announcements

- Quiz 2 available today @ 4pm
- Topics: consumption, debt, and deficits

- Topics today
  1. Overlapping Generations (OLG) model
  2. Debts and Deficits

Overlapping Generations Model

Definition: The building blocks of the OLG growth model are

- Time is discrete and infinite: $t = \{0, 1, 2, ...\}$.
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- Each household is comprised of an individual who lives for two periods, one young and one old.
- Households start life with no financial assets and leave no bequests. They have $\ell_1 = 1$ and $\ell_2 = 0$ units of labor, supplied inelastically.

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- Individual consumption is given by
  1. c_{1t}, consumption of young person in t
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$$C_t := c_{1t}L_t + c_{2t}L_{t-1}$$

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- Equilibrium market clearing conditions:
  1. In the goods/product market, must have

$$Y_t = C_t + I_t$$

  2. Since physical capital is the only savings vehicle, we have

$$K_{t+1} = s_t L_t$$

Overlapping Generations Model

- Example: Find the steady state of the OLG model!
- Step 1: Solve the consumer optimization problem

$$\max_{c_{1t}, c_{2t+1}} \quad \ln c_{1t} + \beta \ln c_{2,t+1}$$

$$\text{s.t.} \quad c_{1t} + s_t = w_t$$

$$c_{2t+1} = (1 + r_{t+1})s_t$$

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Using the MRS condition and the lifetime budget constraint:

$$\begin{aligned} c_{1t}^* &= \frac{w_t}{1+\beta} \\ c_{2t+1}^* &= \frac{\beta}{1+\beta}(1 + r_{t+1}) w_t \\ s_t^* &= \frac{\beta}{1+\beta} w_t \end{aligned}$$

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$$\begin{aligned} w_t &= \frac{\partial}{\partial L_t} Y_t \\ &= f(k_t) - k_t f'(k_t) \\ &= (1-\alpha) k_t^\alpha \end{aligned}$$

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\Rightarrow k_{t+1} &= \frac{\frac{\beta}{1+\beta} w_t L_t}{(1+n)L_t} && \text{since pop growth is } n \\
&= \frac{\frac{\beta}{1+\beta}(1-\alpha)k_t^\alpha}{(1+n)} && \text{by step 2} \\
&= \frac{\beta(1-\alpha)}{(1+\beta)(1+n)} k_t^\alpha
\end{aligned}
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- We say that the economy is dynamically inefficient if $k_* > k_{gold}$.
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- Since we have *too much* capital, saving less could actually increase consumption for everyone in the economy.
- This is pareto improving, and hence the inefficiency.
- What can be done about this? The government can provide another savings vehicle to crowd out private saving... government debt!

Debts and Deficits

Example: let time be discrete and assume output Y_t, government spending G_t, and tax revenue T_t grow at rates g_Y, g_G, and g_T.

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Y_t &= (1+g_Y)Y_{t-1} \\
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c) Write the law of motion for government debt D_t.

$$D_t = (1+r)D_{t-1} + G_{t-1} - T_{t-1}$$

Debts and Deficits

d) Let $d_t := \frac{D_t}{Y_t}$ denote the debt-to-GDP ratio. Under the assumption that the government chooses to run a primary deficit in each period equal to a fixed share of output $x \in (0, 1)$, find the law of motion for d_t.

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&= \frac{1+r}{1+g_Y} d_{t-1} + \frac{x}{1+g_Y}
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We can find a stable solution $d_* = d_t = d_{t-1}$ whenever the coefficient on d_{t-1} is less than 1 in absolute value. That is, when

$$\begin{aligned} \frac{1+r}{1+g_Y} &\leq 1 \\ r &\leq g_Y \end{aligned}$$

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which is satisfied in this case. The steady state value is given by

$$\begin{aligned} d_* &= \frac{1+r}{1+g_Y} d_* + \frac{x}{1+g_Y} \\ d_* &= \frac{x}{g_Y - r} \end{aligned}$$

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We should observe that d_* shrinks with greater output growth. That is

$$\begin{aligned}\frac{\partial}{\partial g_Y} d_* &= \frac{\partial}{\partial g_Y}\left\{x \cdot (g_Y - r)^{-1}\right\} \\ &= -x \cdot (g_Y - r)^{-2} \\ &= -\frac{x}{(g_Y - r)^2} < 0\end{aligned}$$